Quantifying the Closeness to a Set of Random Curves via the Mean Marginal Likelihood

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Motivation: Trajectory Optimization

Aircraft trajectory optimization is a problem that has been extensively studied by the mathematical community, with applications for example in fuel consumption, flight time and noise reduction, as well as in collision avoidance. More recently, a particular framework has been considered in which the aircraft dynamics have been estimated from previous flights data [1, 2]. This setting raises the question of whether the optimized trajectory does not deviate too much from the validity region of the dynamics model, which corresponds to the area occupied by the data used to build it. Moreover, the simulated trajectory is usually wanted in practice to "seem real" for better acceptance by the pilots and Air Traffic Control. These questions may both be addressed by quantifying the closeness between the optimization solution and the set of real flights used to identify the model.

Mean Marginal Likelihood

We suppose that the real flights are observations of the same *functional random variable* $Z = (Z_t)$ valued in $\mathcal{C}(\mathbb{T}, E)$, with E compact subset of $\mathbb{R}^d$ and $\mathbb{T} = [0, t_f]$. We propose to use its marginal densities f_t to evaluate locally the distance of the optimized trajectory $\boldsymbol{y}$ w.r.t. the set of real flights.

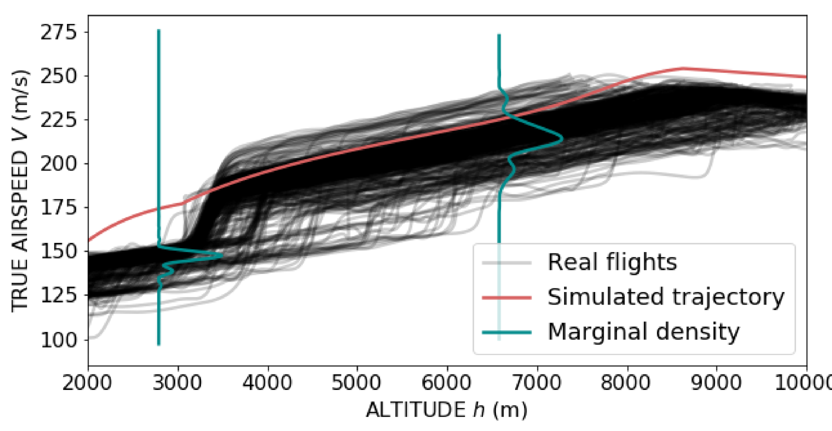

Figure 1: Illustration of the marginal likelihoods.

Up to a certain continuous scaling map

$$\psi : L^1(E, \mathbb{R}_+) \times \mathbb{R} \to [0; 1],$$

we define the *mean marginal likelihood* (MML) as the average over t of the marginal densities f_t evaluated on the points of $\boldsymbol{y}$:

$$\mathrm{MML}(Z, \boldsymbol{y}) = \frac{1}{t_f} \int_0^{t_f} \psi[f_t, \boldsymbol{y}(t)] dt.$$

Possible scalings are the *normalized density* [3]

$$\psi[f_t, \boldsymbol{y}(t)] := \frac{\boldsymbol{y}(t)}{\max_{z \in E} f_t(z)},$$

and the *confidence level* (figure 2)

$$\psi[f_t, \boldsymbol{y}(t)] := \mathbb{P}\left(f_t(Z_t) \leq f_t(\boldsymbol{y}(t))\right).$$

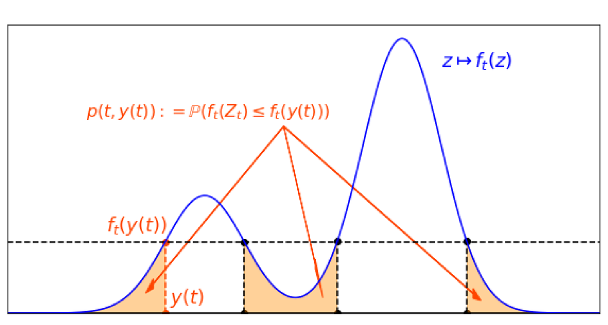

Figure 2: Confidence level for a bimodal distribution

Marginal Likelihood Estimation

In practice, the m trajectories are sampled at variable discrete times:

$$\mathcal{T}^D := \{(t_j^r, z_j^r)\}_{\substack{1 \leq j \leq n \\ 1 \leq r \leq m}} \subset \mathbb{T} \times E, \qquad z_j^r := \boldsymbol{z}^r(t_j^r),$$

$$\mathcal{Y} := \{(\tilde{t}_j, y_j)\}_{j=1}^{\tilde{n}} \subset \mathbb{T} \times E, \qquad y_j := \boldsymbol{y}(\tilde{t}_j).$$

Hence, we approximate the MML using a Riemann sum which aggregates consistent estimators $\hat{f}^m_{\tilde{t}_j}$ of the marginal densities $f_{\tilde{t}_j}$:

$$\mathrm{EMML}_m(\mathcal{T}^D, \mathcal{Y}) := \frac{1}{t_f} \sum_{j=1}^{\tilde{n}} \psi[\hat{f}^m_{\tilde{t}_j}, y_j] \Delta \tilde{t}_j.$$

Marginal density estimation can be done by uniformly partitioning the space of times $\mathbb{T}$ into *bins* and building standard density estimators using the data points whose sampling times fall in each bin (figure 3).

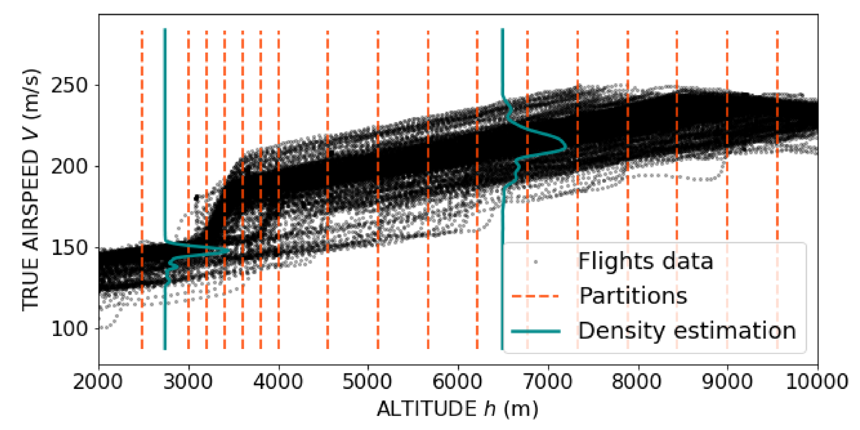

Figure 3: Illustration of the marginal density estimation.

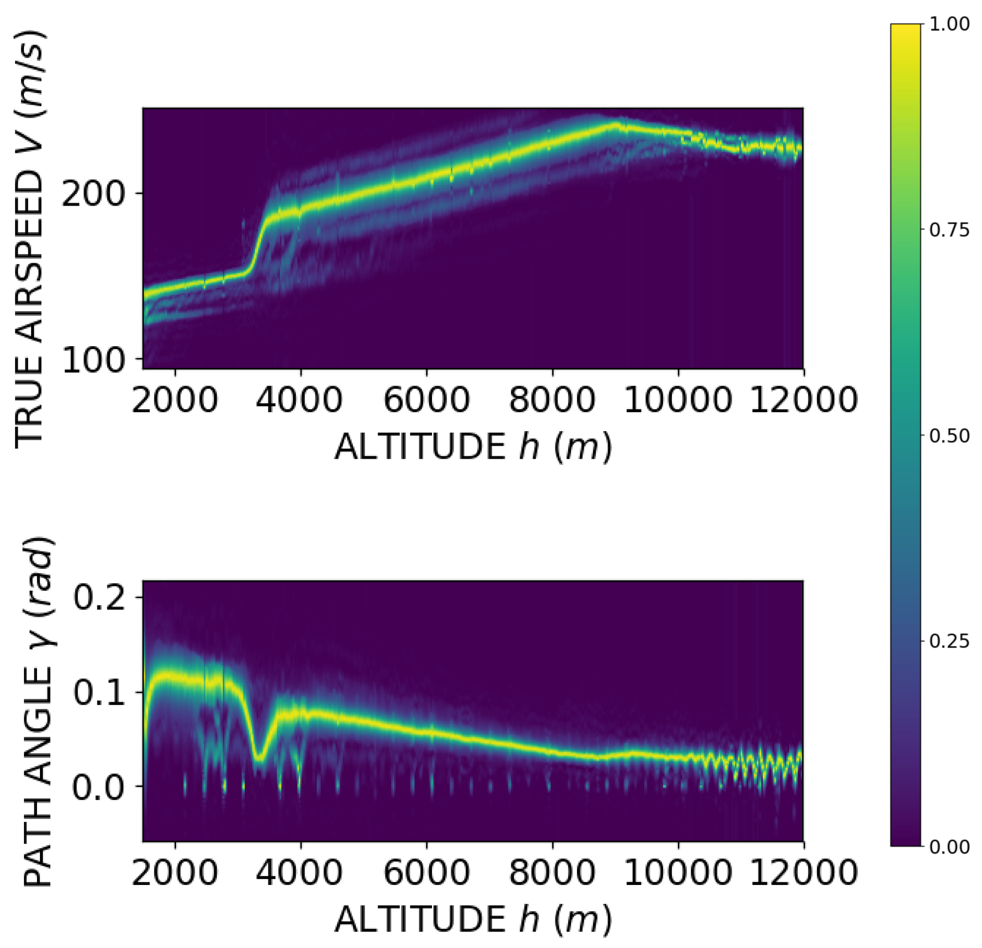

Figure 4: Heatmap of the estimated confidence levels using an adaptive kernel estimator to approximate the marginal likelihoods.

Numerical simulations indicate that the discriminative power of the MML surpasses well-established techniques, such as functional-PCA [4] and least-squares conditional density estimation [5] in our dataset:

Table 1: Average and standard deviation of the likelihood scores obtained using the kernel-MML, GMM-FPCA and integrated LS-CDE for 50 real flights (*Real*), 50 optimized flights with operational constraints (*Opt1*) and 50 optimized flights without constraints (*Opt2*).

Var.	Estimated Likelihoods		
	Real	Opt1	Opt2
MML	0.63 ± 0.07	0.43 ± 0.08	0.13 ± 0.02
FPCA	0.16 ± 0.12	6.4e-03 ± 3.8e-03	3.6e-03 ± 5.4e-03
LS-CDE	0.77 ± 0.05	0.68 ± 0.04	0.49 ± 0.06

Optimal Control Penalization

The local scores obtained by this method can be used not only to assess the optimization solutions, but also to penalize the optimization itself:

$$\min_{(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{X} \times \mathbb{U}} \int_0^{t_f} C(t, \boldsymbol{u}(t), \boldsymbol{x}(t)) dt - \lambda \, \mathrm{MML}(Z, \boldsymbol{x}),$$

$$\text{s.t.} \begin{cases} \dot{\boldsymbol{x}}(t) = \hat{g}(t, \boldsymbol{u}(t), \boldsymbol{x}(t)), & \text{for a.e. } t \in [0, t_f], \\ \Phi(\boldsymbol{x}(0), \boldsymbol{x}(t_f)) \in K_\Phi, \\ c_j(t, \boldsymbol{u}(t), \boldsymbol{x}(t)) \leq 0, & j = 1, \dots, n_c. \end{cases}$$

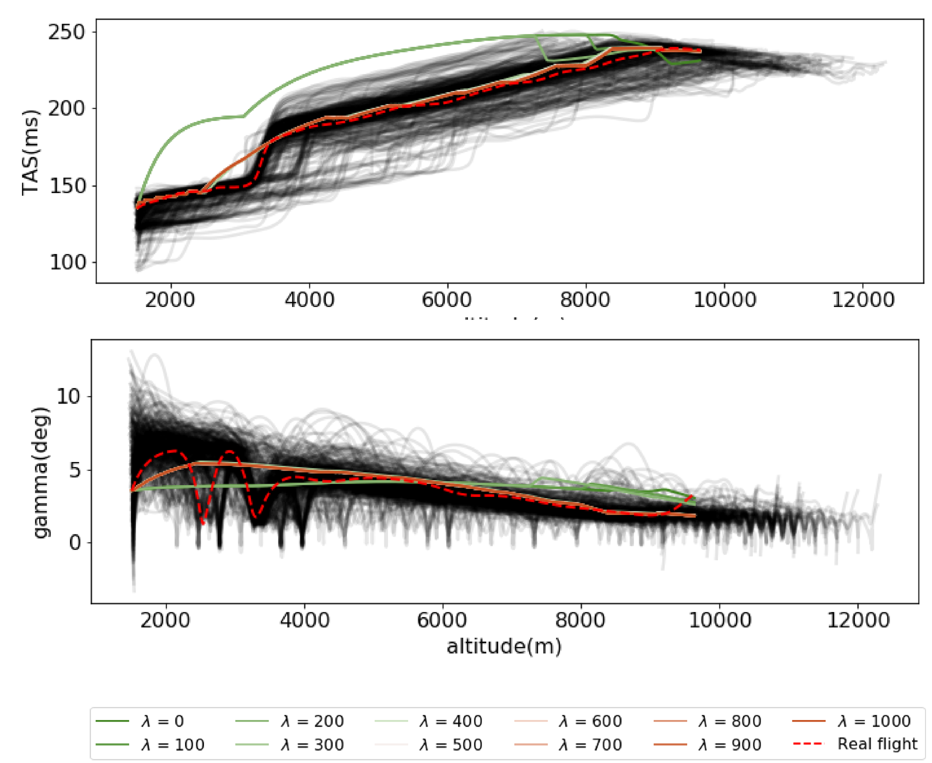

Figure 5: Example of optimized flight with different MML-penalty weights λ.

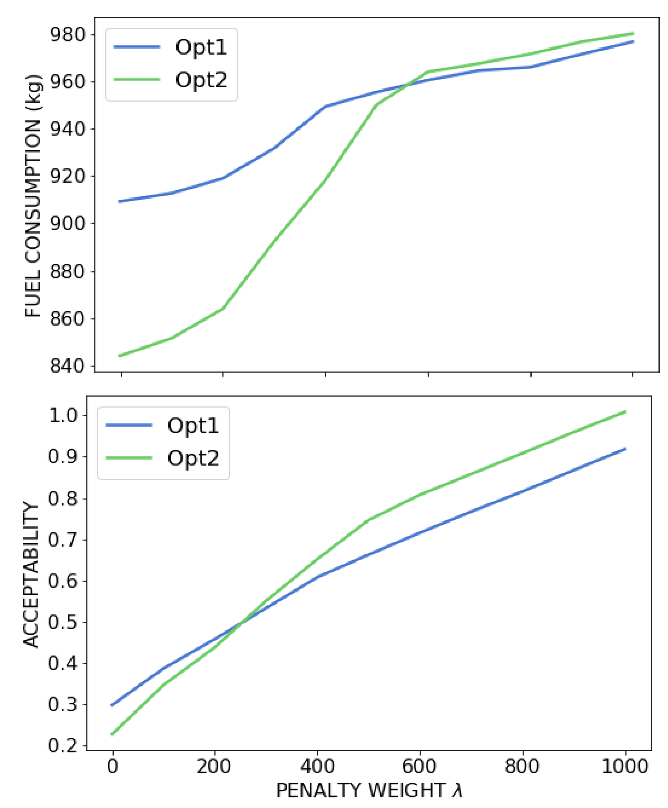

Figure 6: Average over 20 flights of the fuel consumption and MML score (called *acceptability* here) of optimized trajectories with varying MML-penalty weight λ.

References

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