Gaussian Mixture Penalization for Trajectory Optimization Problems

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OPTI CLIMB
More savings, less CO_2

Optimal Control Problem

$$\min_{(\boldsymbol{x},\boldsymbol{u})\in\mathbb{X}\times\mathbb{U}} \int_0^{t_f} C(t,\boldsymbol{u}(t),\boldsymbol{x}(t))dt,$$

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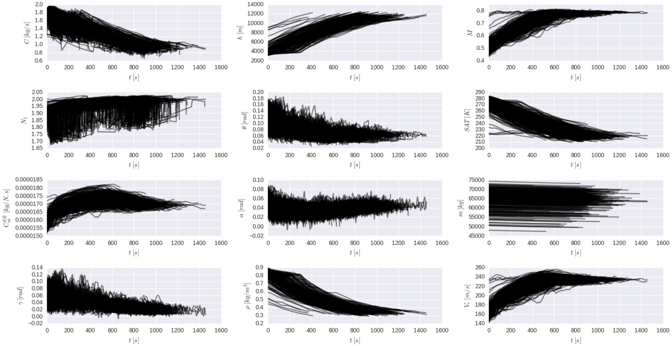

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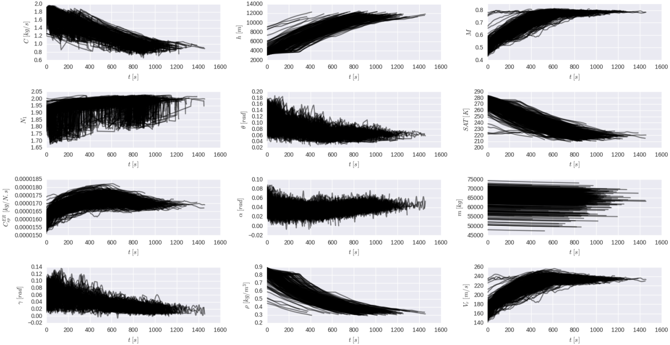

See e.g. [Rommel et al., 2017a] and [Rommel et al., 2017b]

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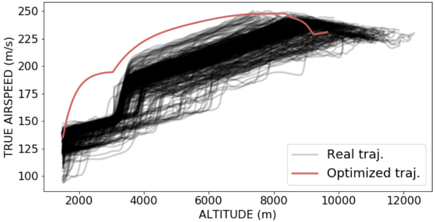

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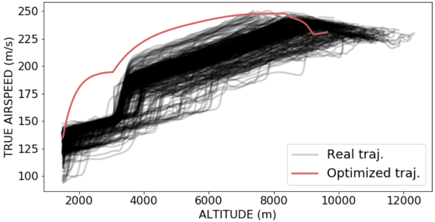

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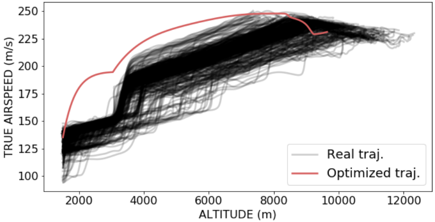

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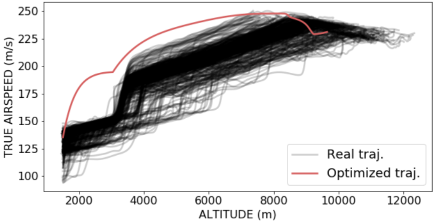

Pilots acceptance Air Traffic Control[1]

How can we quantify the closeness from the optimized trajectory to the set of real flights?

[1]NATS UK air traffic control

Likelihood

Let X be a random variable following an absolutely continuous probability distribution with density function f depending on a parameter θ. Then the function

$$\mathcal{L}(\theta|x) = f_\theta(x) \quad (1)$$

considered as a function of θ, is the likelihood function of *theta*, given the outcome x of X.

[0] Picture source: wikipedia, P-Value, author: Repapetilto CC.

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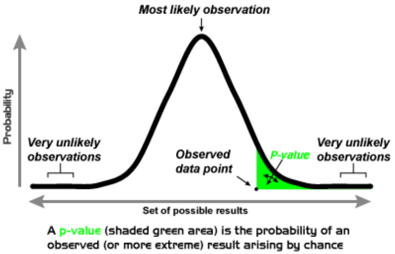

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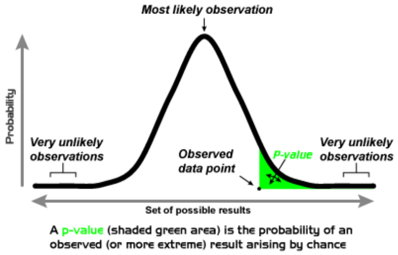

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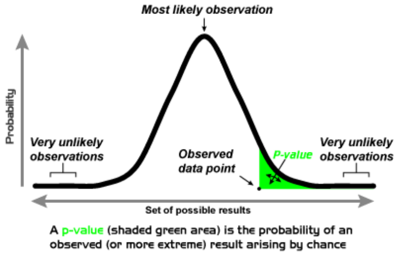

In our case:

- the optimized trajectory plays the role of θ,
- the set of real flights plays the role of x,

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How to apply this to functional data?

Assumption: We suppose that the real flights are observations of the same functional random variable $Z = (Z_t)$ valued in $\mathcal{C}(\mathbb{T}, E)$, with E compact subset of $\mathbb{R}^d$ and $\mathbb{T} = [0, t_f]$.

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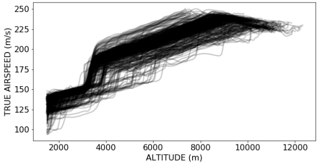

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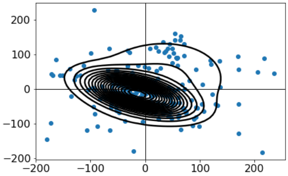

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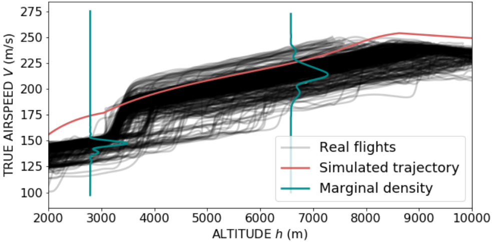
- **Or: we can aggregate the marginal densities**



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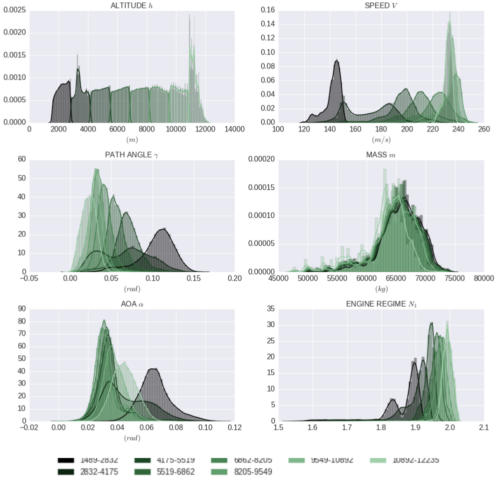

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Marginal densities may have really different shapes

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Mean marginal likelihood [Rommel et al., 2018]

$$\mathrm{MML}(Z, \boldsymbol{y}) = \frac{1}{t_f} \int_0^{t_f} \psi[f_t, \boldsymbol{y}(t)] dt,$$

where $\psi : L^1(E, \mathbb{R}_+) \times \mathbb{R} \to [0; 1]$ is a continuous scaling map.

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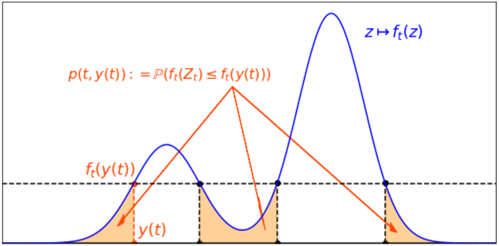
$$\psi[f_t, \boldsymbol{y}(t)] := \frac{\boldsymbol{y}(t)}{\max\limits_{z \in E} f_t(z)},$$

or the confidence level

$$\psi[f_t, \boldsymbol{y}(t)] := \mathbb{P}\left(f_t(Z_t) \leq f_t(\boldsymbol{y}(t))\right).$$



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In practice, the m trajectories are sampled at variable discrete times:

$$\mathcal{T}^D := \{(t_j^r, z_j^r)\}_{\substack{1\leq j\leq n \\ 1\leq r\leq m}} \subset \mathbb{T} \times E, \qquad z_j^r := \boldsymbol{z}^r(t_j^r),$$

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Hence, we approximate the MML using a Riemann sum which aggregates consistent estimators $\hat{f}_{\tilde{t}_j}^m$ of the marginal densities $f_{\tilde{t}_j}$:

$$\boxed{\mathrm{EMML}_m(\mathcal{T}^D, \mathcal{Y}) := \frac{1}{t_f} \sum_{j=1}^{\tilde{n}} \psi[\hat{f}_{\tilde{t}_j}^m, y_j] \Delta \tilde{t}_j.}$$

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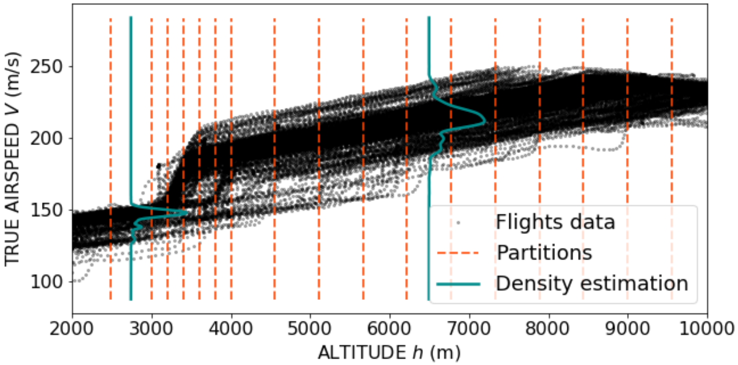
$\Rightarrow$ **Instead, we choose to use a fine partitioning of the time domain.**

Partition based marginal density estimation



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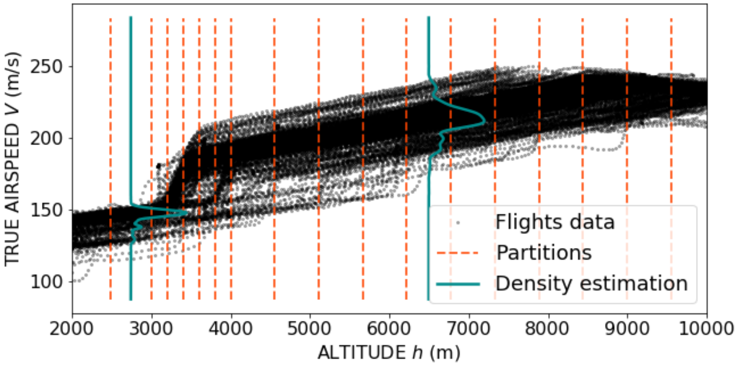

Idea: to average in time the marginal densities over small bins by applying classical multivariate density estimation techniques to each subset.

Consistency

Assumtion 1 - Positive time density

$\nu \in L^\infty(E, \mathbb{R}_+)$ density function of T, s.t.

$$\nu_+ := \operatorname*{ess\,sup}_{t \in \mathbb{T}} \nu(t) < \infty, \qquad \nu_- := \operatorname*{ess\,inf}_{t \in \mathbb{T}} \nu(t) > 0.$$

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Assumtion 2 - Lipschitz in time

Function $(t, z) \in \mathbb{T} \times E \mapsto f_t(z)$ is continuous and

$$|f_{t_1}(z) - f_{t_2}(z)| \leq L|t_1 - t_2|, \qquad L > 0.$$

CONSISTENCY

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ASSUMPTION 3 - SHRINKING BINS

The homogeneous partition $\{B_\ell^m\}_{\ell=1}^{q_m}$ of $[0; t_f]$, with binsize b_m, is s.t.

$$\lim_{m\to\infty} b_m = 0, \qquad \lim_{m\to\infty} m b_m = \infty.$$

Consistency

Assumption 4 - i.i.d. consistency

- $\mathcal{S} = \{(z_k)_{k=1}^N \in E^N : N \in \mathbb{N}^*\}$ set of finite sequences,
- $\Theta : \mathcal{S} \to L^1(E, \mathbb{R}_+)$ multivariate density estimation statistic,
- $\mathcal{G}$ arbitrary family of probability density functions on E, $\rho \in \mathcal{G}$,
- S_ρ^N i.i.d sample of size N drawn from ρ valued in $\mathcal{S}$.

The estimator obtained by applying Θ to S_ρ^N, denoted by

$$\hat{\rho}^N := \Theta[S_\rho^N] \in L^1(E, \mathbb{R}_+),$$

is a (pointwise) consistent density estimator, uniformly in ρ:

For all $z \in E, \varepsilon > 0, \alpha_1 > 0$, there is $N_{\varepsilon,\alpha_1} > 0$ such that, for any $\rho \in \mathcal{G}$,

$$N \geq N_{\varepsilon,\alpha_1} \Rightarrow \mathbb{P}\left(\left|\hat{\rho}^N(z) - \rho(z)\right| < \varepsilon\right) > 1 - \alpha_1.$$

Consistency

We denote by:

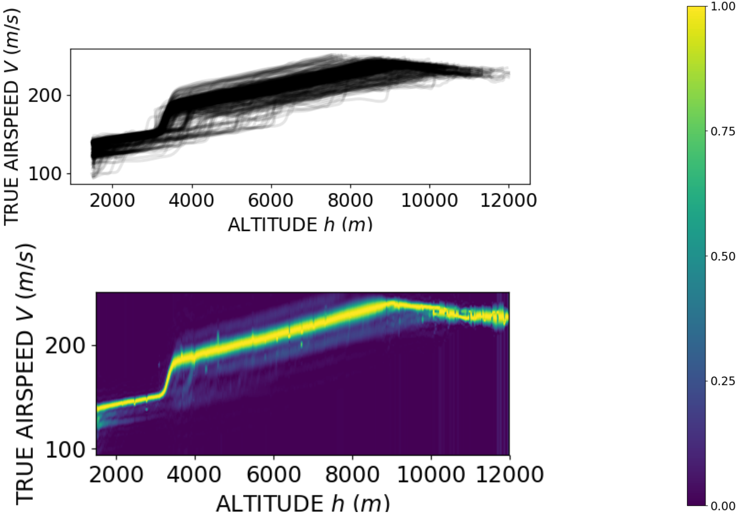
- $\ell^m(t) := \left\lceil \frac{t}{b_m} \right\rceil$ maps time to index of bin containing it;
- $\hat{f}^m_{\ell^m(t)} := \Theta[\mathcal{T}^m_{\ell^m(t)}]$ estimator trained using subset of data points $\mathcal{T}^m_{\ell^m(t)}$ whose sampling times fall in the bin containing t;

Theorem 1 - [Rommel et al., 2018]

Under assumptions 1 to 4, for any $z \in E$ and $t \in \mathbb{T}$, $\hat{f}^m_{\ell^m(t)}(z)$ consistently approximates the marginal density $f_t(z)$ as the number of curves m grows:

$$\forall \varepsilon > 0, \quad \lim_{m \to \infty} \mathbb{P}\left(|\hat{f}^m_{\ell^m(t)}(z) - f_t(z)| < \varepsilon \right) = 1.$$

Marginal density estimation results



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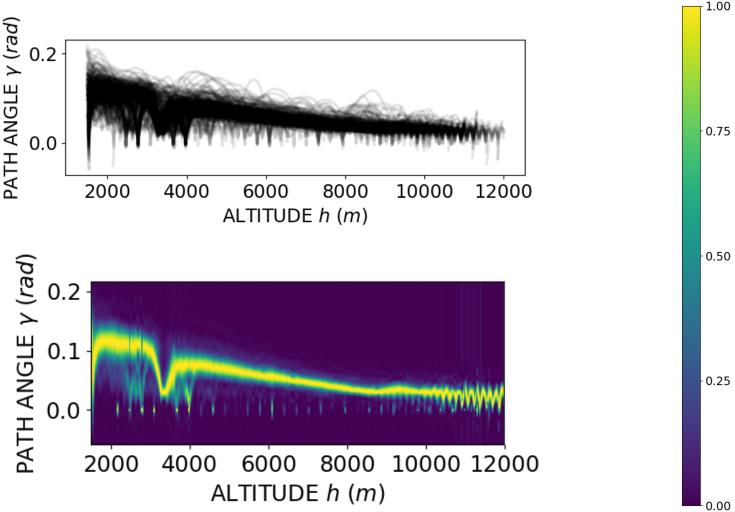

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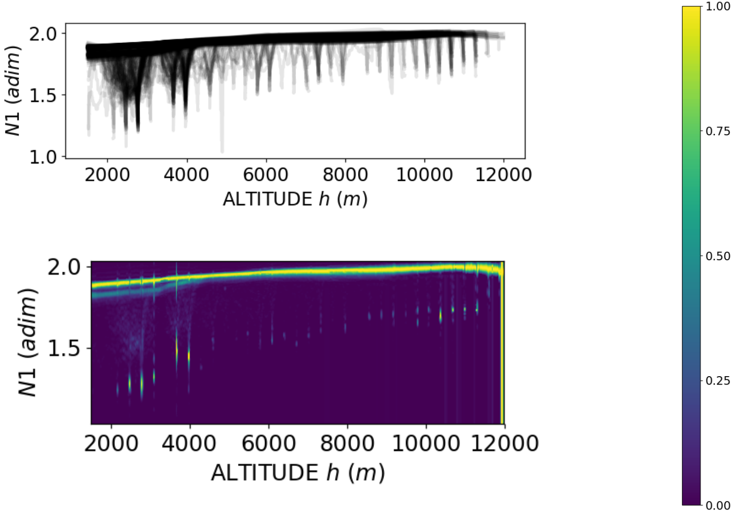

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Var.	Estimated Likelihoods		
	Real	Opt1	Opt2
MML	**0.63** $\pm$ **0.07**	**0.43** $\pm$ **0.08**	**0.13** $\pm$ **0.02**
FPCA	0.16 $\pm$ 0.12	6.4e-03 $\pm$ 3.8e-03	3.6e-03 $\pm$ 5.4e-03
LS-CDE	0.77 $\pm$ 0.05	0.68 $\pm$ 0.04	0.49 $\pm$ 0.06

MML penalty

The MML can be used not only to assess the optimization solutions, but also to penalize the optimization itself:

$$\min_{(\boldsymbol{x},\boldsymbol{u})\in\mathbb{X}\times\mathbb{U}} \int_0^{t_f} C(t,\boldsymbol{u}(t),\boldsymbol{x}(t))dt$$

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- λ sets trade-off between a fuel minimization and a likelihood maximization,
- If (OCP) is solved using NLP techniques, parametric estimator of MML is needed.

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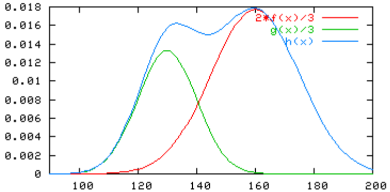

$$\sum_{k=1}^{K} w_{t,k} = 1, \qquad w_{t,k} \geq 0,$$

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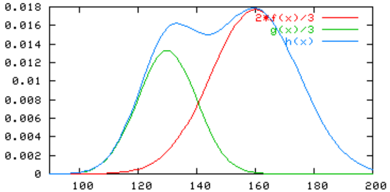

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Assuming that the number of components is known, the weights $w_{t,k}$, means $\mu_{t,k}$ and covariance matrices $\Sigma_{t,k}$ need to be estimated.

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Expectation-Maximization - [Dempster et al., 1977]

Initialization: $\hat{\theta} = (\hat{w}_{t,k}, \hat{\mu}_{t,k}, \hat{\Sigma}_{t,k})_{k=1}^{K} = (w_{t,k}^{0}, \mu_{t,k}^{0}, \Sigma_{t,k}^{0})_{k=1}^{K}$,

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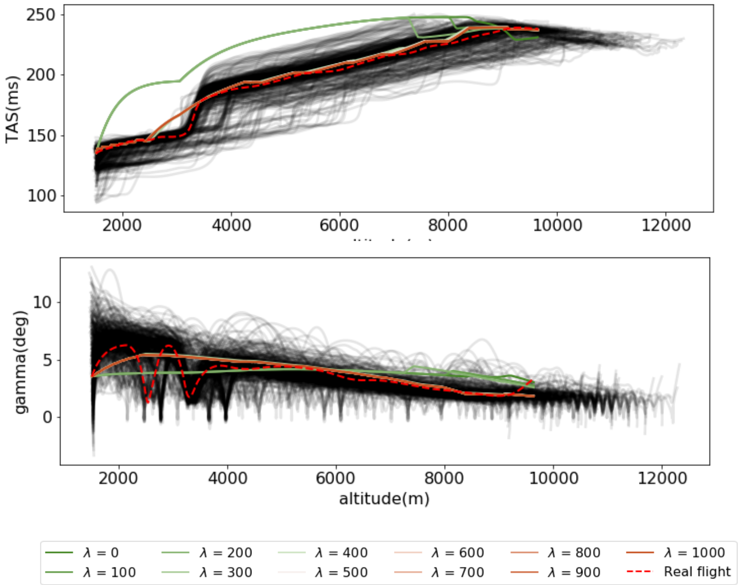
Maximization:

$$\boxed{\hat{\mu}_{t,k} = \frac{\sum_{i=1}^{N} \hat{\pi}_{k,i} z_i}{\sum_{i=1}^{N} \hat{\pi}_{k,i}},} \qquad \boxed{\hat{\Sigma}_{t,k} = \frac{\sum_{i=1}^{N} \hat{\pi}_{k,i} (z_i - \hat{\mu}_{t,k})(z_i - \hat{\mu}_{t,k})^\top}{\sum_{i=1}^{N} \hat{\pi}_{k,i}}.}$$

Penalty effect



Consumption x Acceptability trade-off

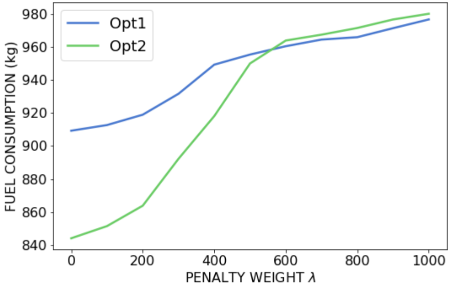


Figure: Average over 20 flights of the fuel consumption and MML score (called acceptability here) of optimized trajectories with varying MML-penalty weight λ.

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$\Rightarrow$ How could we automatically set the trade-off ?...

THANK YOU FOR YOUR ATTENTION !!

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