Quantifying the Closeness to a Set of Random Curves via the Mean Marginal Likelihood

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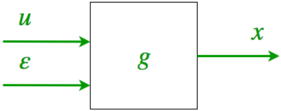
COMPSTAT - August 28^{th} 2018

Motivation - Optimal Control



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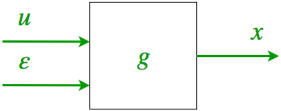

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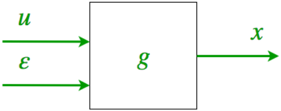

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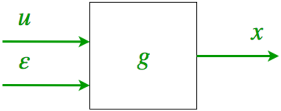

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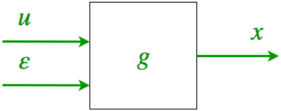

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Use of past data to learn how to control a system efficiently

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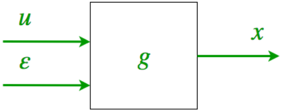

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"Model-based reinforcement learning" - [Recht, 2018]

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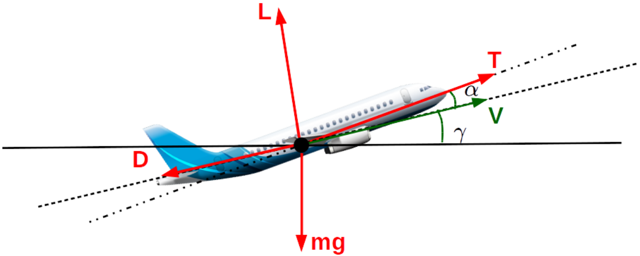

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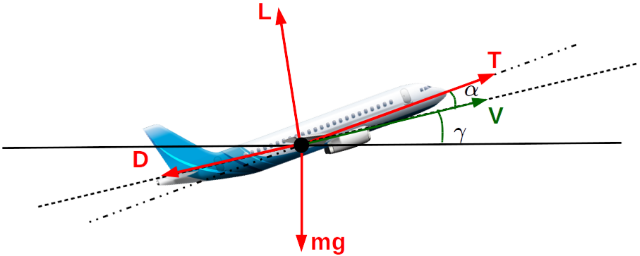

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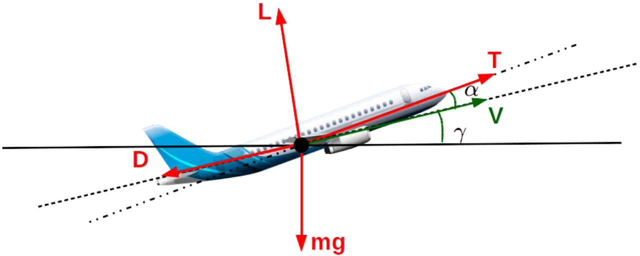

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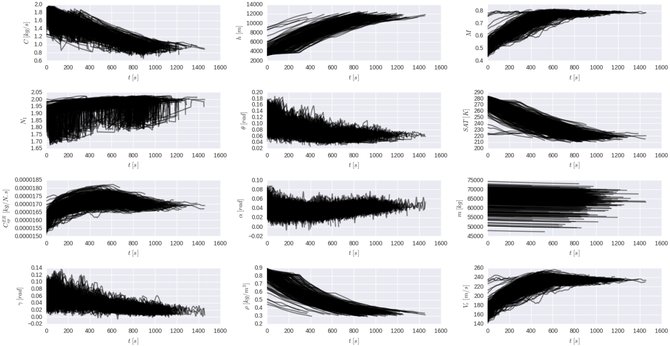

Recorded flights = functional data

Trajectory acceptability

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Does it look like a real trajectory ?

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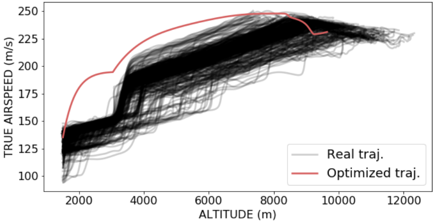

[1]NATS UK air traffic control

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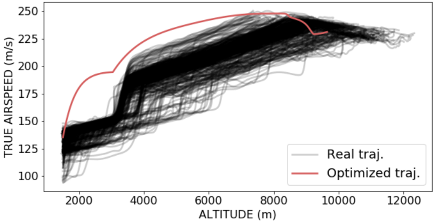

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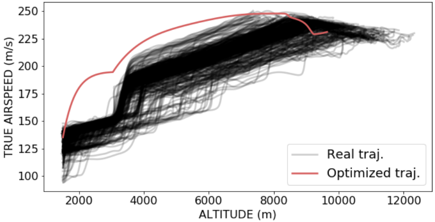

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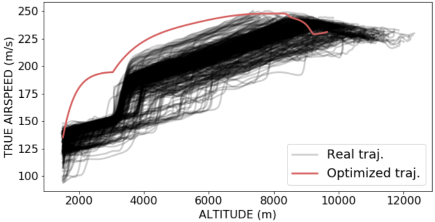

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Air Traffic Control[1]

How can we quantify the closeness from the optimized trajectory to the set of real flights?

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Likelihood

- $X \sim f_{\theta^*}$,
- x: observation of X,
- Likelihood function of θ, given x:

$$\mathcal{L}(\theta|x) = f_\theta(x). \tag{1}$$

[0] Picture source: wikipedia, P-Value, author: Repapetilto CC.

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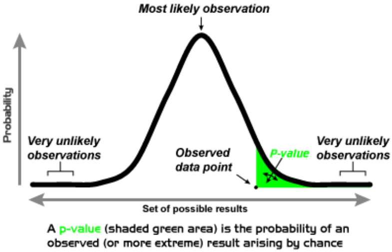

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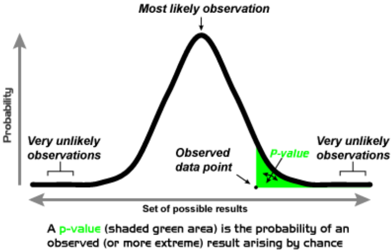

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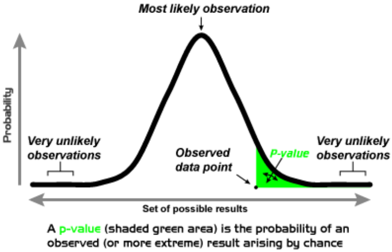

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How to apply this to functional data?

Assumption: We suppose that the real flights are observations of the same functional random variable $Z = (Z_t)$ valued in $\mathcal{C}(\mathbb{T}, E)$, with E compact subset of $\mathbb{R}^d$ and $\mathbb{T} = [0, t_f]$.

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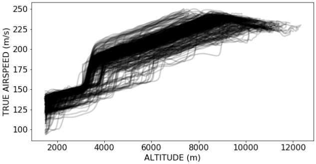

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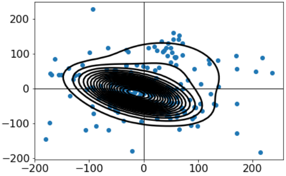

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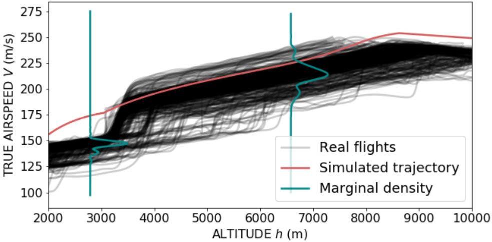
- **Or: we can aggregate the marginal densities**



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Marginal densities may have really different shapes

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Mean marginal likelihood

$$\mathrm{MML}(Z, \boldsymbol{y}) = \frac{1}{t_f} \int_0^{t_f} \psi[f_t, \boldsymbol{y}(t)] dt,$$

where $\psi : L^1(E, \mathbb{R}_+) \times \mathbb{R} \to [0; 1]$ is a continuous scaling map.

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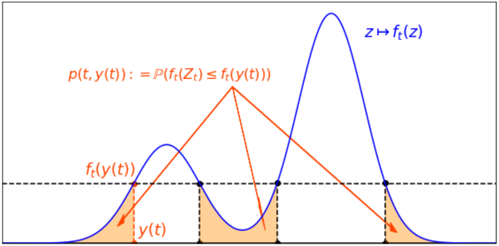
$$\psi[f_t, \boldsymbol{y}(t)] := \frac{\boldsymbol{y}(t)}{\max\limits_{z \in E} f_t(z)},$$

or the confidence level

$$\psi[f_t, \boldsymbol{y}(t)] := \mathbb{P}\left(f_t(Z_t) \leq f_t(\boldsymbol{y}(t))\right).$$



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In practice, the m trajectories are sampled at variable discrete times:

$$\mathcal{T}^D := \{(t_j^r, z_j^r)\}_{\substack{1\leq j\leq n \\ 1\leq r\leq m}} \subset \mathbb{T} \times E, \qquad z_j^r := \boldsymbol{z}^r(t_j^r),$$

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Hence, we approximate the MML using a Riemann sum which aggregates consistent estimators $\hat{f}_{\tilde{t}_j}^m$ of the marginal densities $f_{\tilde{t}_j}$:

$$\boxed{\mathrm{EMML}_m(\mathcal{T}^D, \mathcal{Y}) := \frac{1}{t_f} \sum_{j=1}^{\tilde{n}} \psi[\hat{f}_{\tilde{t}_j}^m, y_j] \Delta \tilde{t}_j.}$$

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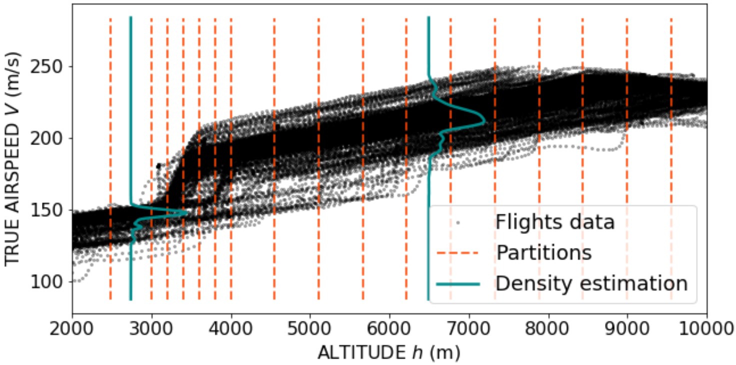
$\Rightarrow$ We could apply SOA conditional density estimation techniques, such as LS-CDE [Sugiyama et al., 2010],

$\Rightarrow$ **Instead, we choose to use a fine partitioning of the time domain.**

Partition based marginal density estimation



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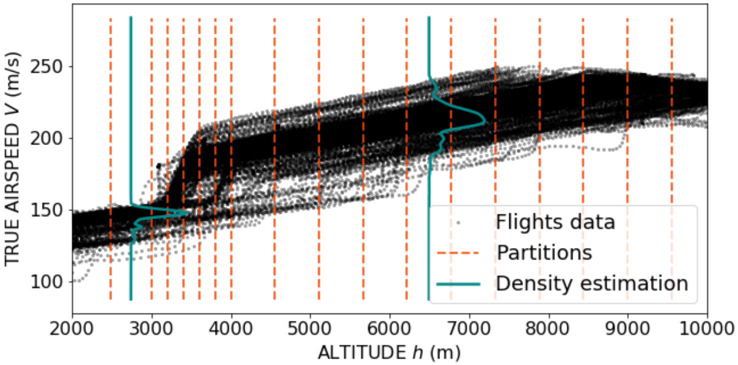

Idea: to average in time the marginal densities over small bins by applying classical multivariate density estimation techniques to each subset.

Consistency

We denote by:

- $\Theta : \mathcal{S} \to L^1(E, \mathbb{R}_+)$ multivariate density estimation statistic,
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- m the number of random curves;
- $\mathcal{T}_t^m$ subset of data points whose sampling times fall in the bin containing t;
- $\hat{f}_t^m := \Theta[\mathcal{T}_t^m]$ estimator trained using $\mathcal{T}_t^m$.

Consistency

Assumtion 1 - Positive time density

$\nu \in L^\infty(E, \mathbb{R}_+)$ density function of T, s.t.

$$\nu_+ := \operatorname*{ess\,sup}_{t \in \mathbb{T}} \nu(t) < \infty, \qquad \nu_- := \operatorname*{ess\,inf}_{t \in \mathbb{T}} \nu(t) > 0.$$

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Assumption 3 - Shrinking bins

The homogeneous partition $\{B_\ell^m\}_{\ell=1}^{q_m}$ of $[0; t_f]$, with binsize b_m, is s.t.

$$\lim_{m \to \infty} b_m = 0, \qquad \lim_{m \to \infty} m b_m = \infty.$$

Consistency

Assumption 4 - i.i.d. consistency

- $\mathcal{G}$ arbitrary family of probability density functions on E, $\rho \in \mathcal{G}$,
- S_ρ^N i.i.d sample of size N drawn from ρ valued in $\mathcal{S}$.

The estimator obtained by applying Θ to S_ρ^N, denoted by

$$\hat{\rho}^N := \Theta[S_\rho^N] \in L^1(E, \mathbb{R}_+),$$

is a (pointwise) consistent density estimator, uniformly in ρ:

For all $z \in E, \varepsilon > 0, \alpha_1 > 0$, there is $N_{\varepsilon,\alpha_1} > 0$ such that, for any $\rho \in \mathcal{G}$,

$$N \geq N_{\varepsilon,\alpha_1} \Rightarrow \mathbb{P}\left(\left|\hat{\rho}^N(z) - \rho(z)\right| < \varepsilon\right) > 1 - \alpha_1.$$

Consistency

Theorem 1 - [Rommel et al., 2018]

Under assumptions 1 to 4, for any $z \in E$ and $t \in \mathbb{T}$, $\hat{f}^m_{\ell^m(t)}(z)$ consistently approximates the marginal density $f_t(z)$ as the number of curves m grows:

$$\forall \varepsilon > 0, \quad \lim_{m \to \infty} \mathbb{P}\left(|\hat{f}^m_t(z) - f_t(z)| < \varepsilon \right) = 1.$$

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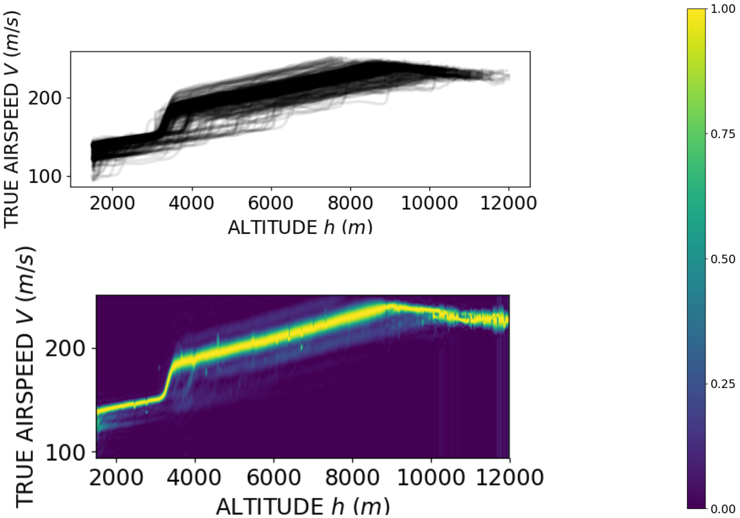
- $m \to \infty \neq N \to \infty$,
- Number of samples $=$ random.

Marginal density estimation results



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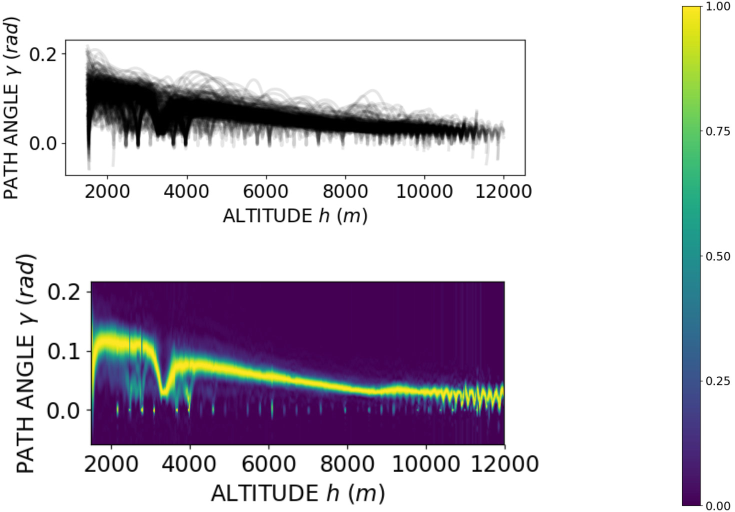

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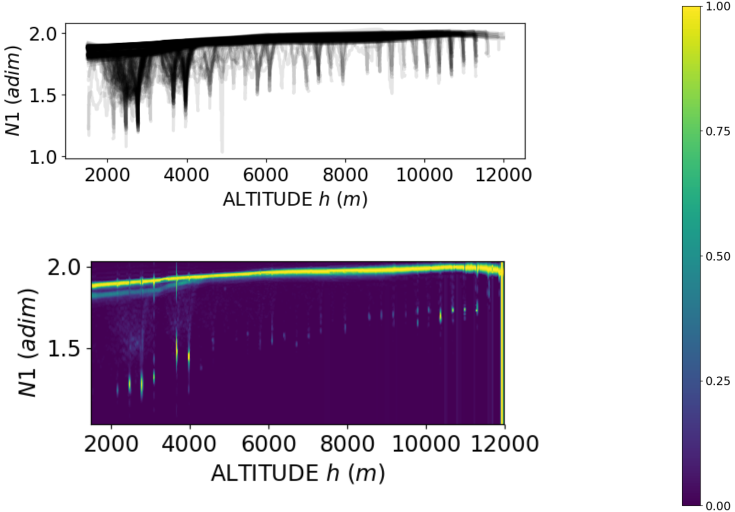

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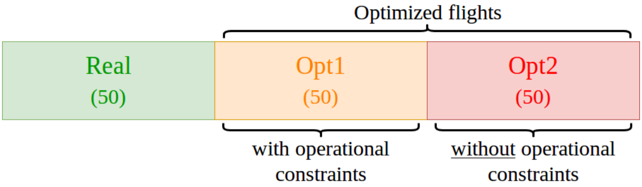

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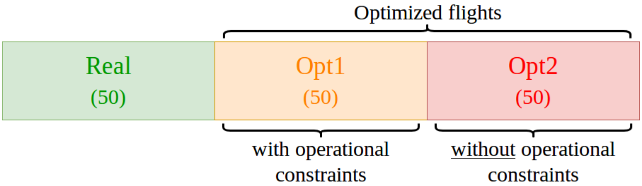

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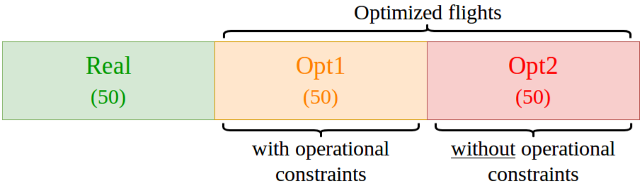

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Var.	Estimated Likelihoods		
	Real	Opt1	Opt2
MML	**0.63 ± 0.07**	**0.43 ± 0.08**	**0.13 ± 0.02**
FPCA	0.16 ± 0.12	6.4E-03 ± 3.8E-03	3.6E-03 ± 5.4E-03
LS-CDE	0.77 ± 0.05	0.68 ± 0.04	0.49 ± 0.06

MML PENALTY

The MML can be used not only to assess the optimization solutions, but also to penalize the optimization itself:

$$\min_{(\boldsymbol{x},\boldsymbol{u})\in\mathbb{X}\times\mathbb{U}} \int_0^{t_f} C(\boldsymbol{u}(t),\boldsymbol{x}(t))dt$$

$$\text{s.t.} \quad \begin{cases} \dot{\boldsymbol{x}}(t) = g(\boldsymbol{u}(t),\boldsymbol{x}(t)) + \varepsilon(t), & \text{for a.e. } t\in[0,t_f], \\ \text{Other constraints...} \end{cases} \tag{OCP}$$

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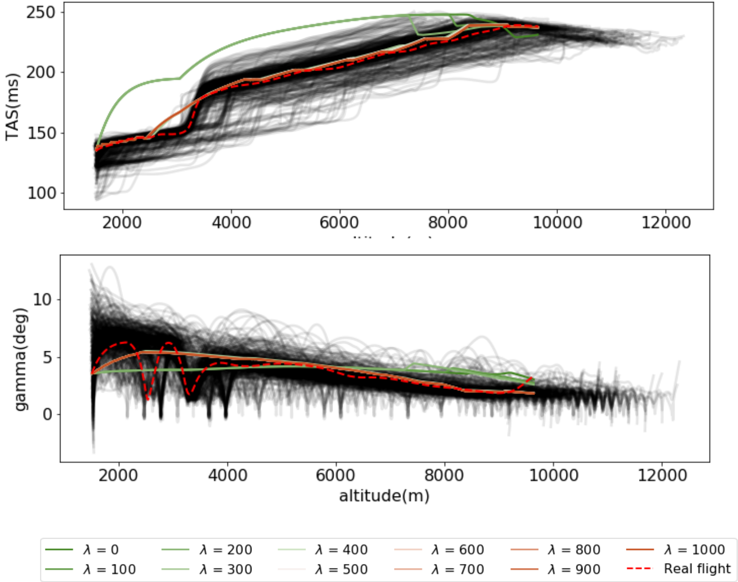
- λ sets trade-off between a fuel minimization and a likelihood maximization,

Penalty effect



Consumption x Acceptability trade-off

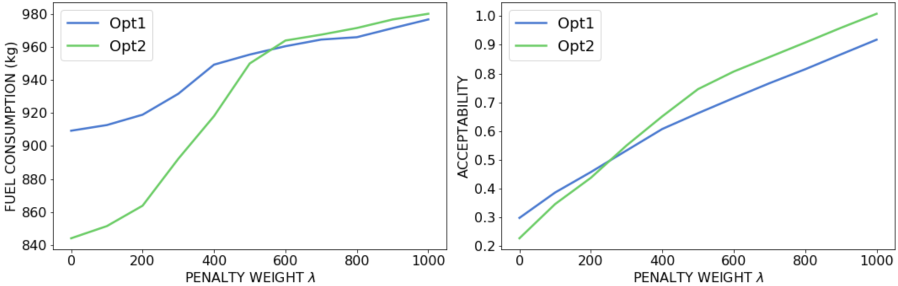


Figure: Average over 20 flights of the fuel consumption and MML score (called acceptability here) of optimized trajectories with varying MML-penalty weight λ.

Conclusion

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3. Applicable to the case of aircraft climb trajectories,
   - Competitive with other well-established SOA approaches,
4. Showed that it can be used in optimal control problems to obtain solutions close to optimal, and still realistic.

THANK YOU FOR YOUR ATTENTION

References

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